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8-Step Method for Third Order Fredholm Integro-Differential Equations with Vieta–Pell-Lucas as Approximating Function.

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Abstract

Purpose: In this paper, we derive multistep numerical methods for third order linear Fredholm integro-differential equation using Vieta-Pell-Lucas polynomial as approximating function.

Methodology: These new method is implemented by embedding it with Trapezoidal, Booles and Simpson 1/3 numerical methods. The derivation led to a block of twenty-four discrete schemes which simultaneously solves the third order linear integro-differential equation. The qualitative properties of the schemes have been investigated.

Findings: The analysis of the method reveals that the proposed method is of order seven and the method is found to be consistent and stable, hence convergent.

Unique contributor to theory, policy and practice: Numerical experiments have been performed on some selected problems and the results show that the proposed method perform creditably well when compared with the exact solution of the selected examples.

Keywords: *Booles, Fredholm , Integrodifferential equation, Simpson, Trapezoidal, Vieta-Pell-Lucas.*

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1.0 Introduction

Numerical analysis has been an active and dominant branch of applied Mathematics for a long time with differential and Integral equations at the heart of this field of study. Integral equation (IE) is one of the vital tools for modelling real problems in various areas of applied mathematics and Engineering.

According to [1], Integral Equations occur naturally in many fields of Sciences and Engineering such as Mechanics, potential theory, Geophysics, Communication theory, Hereditary phenomena in Physics and Biology, Mathematical economics, Queuing theory, Medicine, Acoustics fluid mechanics, among others.

Some numerical methods have been developed to solve different types of Integro-differential equations in the past, such methods include: Adomian decomposition method [2], Compact finite difference method [3], Standard integral collocation approximation method [4], Quadrature-difference method [5,6]. Approximate method proposed in [9], Implicit Runge-Kutta method [10] and linear multistep methods [11] to mention few. Other methods reviewed are [7], in which numerical methods for second order Volterra integro-differential equation was presented using shifted Legendre polynomial with trapezoidal rule. In [8], efficiency of Simpson's 1/3 and Simpson's 3/8 rules was compared for first order Volterra integrodifferential equations using shifted Legendre polynomials as basis function while [20] proposed a Discrete Adomian Decomposition method with Trapezoidal and Simpson's rules for solving Integro-Differential equation.

All the methods reviewed proved to be efficient and suitable in their respective areas of application as provided by the result but not much have been done so far on the application of these methods to third order Fredholm Integrodifferential Equation (FIDEs).

In this paper, we discuss a special numerical method for solving third order Fredholm integrodifferential equation of the form:

$$y'''(x) = f(x) + \lambda \int_{\alpha}^{\beta} \phi(x,t) y(t) dt, \; y^{j}(a) = b_j, \; j = 0,1,2 \qquad (1.0)$$

Interpolation and collocation method is adapted for solving the differential part while the integral part is approximated by using Trapezoidal, Booles and Simpson's 1/3 quadrature formulae respectively.

2.0 Methodology

Many numerical analysts have engaged themselves with derivation of numerical methods for nth order initial value problems for ordinary equations such as standard finite difference method in [12], non-standard finite difference method [13] and linear multistep method [14,15] etc.

A numerical method is presented for third order Linear Fredholm Integro-differential Equation (LFIDE) of the form (1.0) using approximating function of Vieta-Pell-Lucas Polynomials of the form

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$$y(x)=\sum_{i=0}^{k} a_i s_i(x) \qquad (2.0)$$

Where a_i's are real undetermined coefficients and $s_i(x)$'s are parameters of Vieta-Pell-Lucas given by recurrence formula $S_i(x)=2xS_{i-1}(x)-S_{i-2}(x)$,where $i\geq 2$, $S_0(x)=2$ and $S_1(x)=2x, k=9$.

2.1 Derivation of the Method

The method is derived by interpolating (2.0) at x_{n+j} , $j=0,2,4,6,7$ and collocating the third derivative of (2.0) at grid points x_{n+j} , $j=1,2,3,5,7$ which yield a system of interpolation and collocation equations. The resulting system of equations are solved using any suitable method via MAPLE 18 software in order to obtain the values of unknown $a_i, 0\leq i\leq 9$.

In sequel, a continuous scheme of the form

$$y(x)=\sum_{i=0}^{k}\alpha_i(x)y_{n+i}+h^3\sum_{i=0}^{k}\beta_i(x)f(x_{n+i},y(x_{n+i},z(x_{n+i}))) \qquad (2.1)$$

where h is the step size and

$$z(x_{n+i})=h\sum\alpha_i(x)w_{ni}\tau(x_n,x_i,y_i) \qquad (2.2)$$

is the embedded weight function of Boole, Trapezoidal and Simpson 1/3 quadrature formulae obtained.

Consequently, the continuous scheme of the form (2.1) is evaluated at points $x_{n+1}, x_{n+3}, x_{n+5}, x_{n+8}$ and simplified in order to obtain the first four discrete schemes

$$\left.\begin{aligned}
y_{n+1}&=\left(\frac{5173789}{24210688}f_{n+1}+\frac{414405}{1513168}f_{n+2}+\frac{6469293}{24210688}f_{n+3}+\frac{3186807}{24210688}f_{n+5}+\frac{85631}{24210688}f_{n+7}\right)h^3+\frac{4955625}{12105344}y_n+\frac{8039349}{12105344}y_{n+2}-\frac{960485}{12105344}y_{n+4}+\frac{627399}{12105344}y_{n+6}-\frac{4348}{94573}y_{n+7}\\
y_{n+3}&=\left(-\frac{947921}{12105344}f_{n+1}-\frac{992453}{3782920}f_{n+2}-\frac{3705153}{12105344}f_{n+3}-\frac{814739}{12105344}f_{n+5}-\frac{100487}{60526720}f_{n+7}\right)h^3-\frac{877717}{60526720}y_n+\frac{4852431}{60526720}y_{n+2}+\frac{2074593}{60526720}y_{n+4}-\frac{131227}{60526720}y_{n+6}+\frac{2103}{94573}y_{n+7}\\
y_{n+5}&=\left(\frac{1932287}{24210688}f_{n+1}+\frac{398823}{1513168}f_{n+2}+\frac{10626799}{24210688}f_{n+3}-\frac{1544099}{24210688}f_{n+5}+\frac{27125}{24210688}f_{n+7}\right)h^3+\frac{1777459}{12105344}y_n-\frac{6763849}{12105344}y_{n+2}+\frac{14139065}{12105344}y_{n+4}+\frac{3170525}{12105344}y_{n+6}-\frac{1702}{94573}y_{n+7}\\
y_{n+8}&=\left(\frac{65969}{189146}f_{n+1}+\frac{515704}{476825}f_{n+2}+\frac{385369}{189146}f_{n+3}-\frac{84957}{189146}f_{n+5}+\frac{488287}{945730}f_{n+7}\right)h^3+\frac{59602}{94573}y_n-\frac{218969}{94573}y_{n+2}+\frac{344207}{94573}y_{n+4}-\frac{449563}{94573}y_{n+6}+\frac{359296}{94573}y_{n+7}
\end{aligned}\right\} \qquad (2.3)$$

Evaluating the first derivative of the continuous scheme of the form (2.1) at all the grid points, we obtain the next eight discrete schemes:

$$\left.\begin{aligned}
&y'_n(x)=\Big(\frac{35683961}{22697520}f_{n+1}+\frac{934219}{1418595}f_{n+2}+\frac{39799499}{7565840}f_{n+3}+\frac{65675003}{22697520}f_{n+5}+\frac{336815}{4539504}f_{n+7}\Big)h^2+\frac{1}{h}\Big(\frac{1349003}{15888264}y_n-\frac{4304139}{3782920}y_{n+2}+\frac{780493}{756584}y_{n+4}+\frac{2269225}{2269752}y_{n+6}\\
&-\frac{3238864}{3310055}y_{n+7}\Big)\\
&y'_{n+1}(x)=\Big(-\frac{438957101}{1694748160}f_{n+1}-\frac{602073}{21184352}f_{n+2}-\frac{1030335837}{1694748160}f_{n+3}-\frac{576013863}{1694748160}f_{n+5}-\frac{14700559}{1694748160}f_{n+7}\Big)h^2+\frac{1}{h}\Big(-\frac{101489805}{169474816}y_n+\frac{91222677}{121053440}y_{n+2}-\frac{11134723}{72632064}y_{n+4}-\frac{2808837}{24210688}y_{n+6}\\
&+\frac{2275891}{19860330}y_{n+7}\Big)\\
&y'_{n+2}(x)=\Big(-\frac{4754059}{31776528}f_{n+1}-\frac{822445}{1986033}f_{n+2}-\frac{8227043}{31776528}f_{n+3}-\frac{2477137}{31776528}f_{n+5}-\frac{65281}{31776528}f_{n+7}\Big)h^2+\frac{1}{h}\Big(-\frac{4307405}{15888264}y_n+\frac{201441}{3782920}y_{n+2}+\frac{499895}{2269752}y_{n+4}-\frac{66355}{2269752}y_{n+6}\\
&+\frac{266608}{9930165}y_{n+7}\Big)\\
&y'_{n+3}(x)=\Big(\frac{5724531}{1694748160}f_{n+1}-\frac{8109}{105921760}f_{n+2}-\frac{220179453}{1694748160}f_{n+3}+\frac{4949433}{1694748160}f_{n+5}-\frac{30447}{1694748160}f_{n+7}\Big)h^2+\frac{1}{h}\Big(\frac{2762761}{508424448}y_n-\frac{62509563}{121053440}y_{n+2}+\frac{12506319}{24210688}y_{n+4}-\frac{423007}{72632064}y_{n+6}\\
&+\frac{1377}{6620110}y_{n+7}\Big)\\
&y'_{n+4}(x)=\Big(\frac{19682153}{158882640}f_{n+1}+\frac{1371201}{3319955}f_{n+2}+\frac{33799067}{52960880}f_{n+3}+\frac{2809753}{52960880}f_{n+5}+\frac{331579}{158882640}f_{n+7}\Big)h^2+\frac{1}{h}\Big(\frac{1211523}{5296088}y_n-\frac{3503229}{3782920}y_{n+2}+\frac{1480429}{2269752}y_{n+4}+\frac{54693}{756584}y_{n+6}\\
&-\frac{331579}{158882640}y_{n+7}\Big)\\
&y'_{n+5}(x)=\Big(-\frac{10137031}{5084244480}f_{n+1}-\frac{1939799}{317765280}f_{n+2}-\frac{42704087}{5084244480}f_{n+3}-\frac{215872871}{1694748160}f_{n+5}+\frac{581815}{1016848896}f_{n+7}\Big)h^2+\frac{1}{h}\Big(-\frac{259777}{72632064}y_n+\frac{2242677}{121053440}y_{n+2}-\frac{38982947}{72632064}y_{n+4}+\frac{39407185}{72632064}y_{n+6}\\
&-\frac{58987}{2837190}y_{n+7}\Big)\\
&y'_{n+6}(x)=\Big(-\frac{17307089}{158882640}f_{n+1}-\frac{730355}{1986033}f_{n+2}-\frac{30524531}{52960880}f_{n+3}+\frac{48781693}{158882640}f_{n+5}-\frac{945811}{158882640}f_{n+7}\Big)h^2+\frac{1}{h}\Big(-\frac{3206155}{15888264}y_n+\frac{2968611}{3782920}y_{n+2}-\frac{1036453}{756584}y_{n+4}+\frac{1358647}{2269752}y_{n+6}\\
&+\frac{623536}{3310055}y_{n+7}\Big)\\
&y'_{n+7}(x)=\Big(\frac{23571859}{145264128}f_{n+1}+\frac{1745265}{3026336}f_{n+2}+\frac{37670881}{48421376}f_{n+3}-\frac{40895821}{48421376}f_{n+5}+\frac{7268081}{145264128}f_{n+7}\Big)h^2+\frac{1}{h}\Big(\frac{51804195}{169474816}y_n-\frac{151374363}{121053440}y_{n+2}+\frac{159441485}{72632064}y_{n+4}-\frac{84082005}{24210688}y_{n+6}\\
&+\frac{44140291}{19860330}y_{n+7}\Big)\\
&y'_{n+8}(x)=\Big(\frac{89751139}{158882640}f_{n+1}+\frac{4033923}{3310055}f_{n+2}+\frac{737460443}{158882640}f_{n+3}+\frac{687660377}{158882640}f_{n+5}+\frac{242418217}{158882640}f_{n+7}\Big)h^2+\frac{1}{h}\Big(\frac{14652103}{15888264}y_n-\frac{9008559}{3782920}y_{n+2}+\frac{4499891}{2269752}y_{n+4}-\frac{3395851}{2269752}y_{n+6}\\
&+\frac{9659728}{9930165}y_{n+7}\Big)
\end{aligned}\right\}$$

(2.4)

Similarly, evaluating the second derivative of the continuous scheme of the form (2.1) at all points leads to another set of eight discrete schemes:

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$$y''_{n}(x)=(-\frac{5485658497}{953295840}f_{n+1}-\frac{78406043}{59580990}f_{n+2}-\frac{8333400163}{317765280}f_{n+3}-\frac{14043357331}{953295840}f_{n+5}-\frac{72163687}{190659168}f_{n+7})h+\frac{1}{h^{2}}(-\frac{126857777}{31776528}y_{n}+\frac{15670955}{1513168}y_{n+2}-\frac{28338605}{4539504}y_{n+4}$$
$$-\frac{23190733}{4539504}y_{n+6}+\frac{9904568}{1986033}y_{n+7})$$

$$y''_{n+1}(x)=(-\frac{45580847}{953295840}f_{n+1}-\frac{7931359}{13240220}f_{n+2}+\frac{408406477}{317765280}f_{n+3}+\frac{224741033}{317765280}f_{n+5}+\frac{3287017}{190659168}f_{n+7})h+\frac{1}{h^{2}}(\frac{10022609}{21184352}y_{n}-\frac{3276445}{3026336}y_{n+2}+\frac{1907653}{3026336}y_{n+4}$$
$$+\frac{636657}{3026336}y_{n+6}-\frac{153052}{662011}y_{n+7})$$

$$y''_{n+2}(x)=(\frac{40650611}{317765280}f_{n+1}+\frac{105883}{11916198}f_{n+2}-\frac{123502453}{317765280}f_{n+3}-\frac{69914981}{953295840}f_{n+5}-\frac{394951}{317765280}f_{n+7})h+\frac{1}{h^{2}}(\frac{2278683}{10592176}y_{n}-\frac{609059}{1513168}y_{n+2}+\frac{255903}{1513168}y_{n+4}$$
$$-\frac{1097}{1513168}y_{n+6}+\frac{12568}{662011}y_{n+7})$$

$$y''_{n+3}(x)=(\frac{170962193}{953295840}f_{n+1}+\frac{76797017}{11916198}f_{n+2}+\frac{256705277}{317765280}f_{n+3}+\frac{212547899}{953295840}f_{n+5}+\frac{4734781}{953295840}f_{n+7})h+\frac{1}{h^{2}}(\frac{21373379}{63553056}y_{n}-\frac{2219981}{3026336}y_{n+2}+\frac{3852767}{9079008}y_{n+4}$$
$$+\frac{368035}{9079008}y_{n+6}-\frac{134356}{1986033}y_{n+7})$$

$$y''_{n+4}(x)=(\frac{6876103}{953295840}f_{n+1}+\frac{9209}{1324022}f_{n+2}+\frac{68546677}{317765280}f_{n+3}-\frac{65327977}{317765280}f_{n+5}-\frac{1616683}{953295840}f_{n+7})h+\frac{1}{h^{2}}(\frac{110125}{10592176}y_{n}+\frac{317347}{1513168}y_{n+2}-\frac{663559}{1513168}y_{n+4}$$
$$+\frac{293817}{1513168}y_{n+6}+\frac{16040}{662011}y_{n+7})$$

$$y''_{n+5}(x)=(-\frac{65675749}{317765280}f_{n+1}-\frac{80167511}{119161980}f_{n+2}-\frac{374946163}{317765280}f_{n+3}+\frac{102906619}{953295840}f_{n+5}-\frac{3869}{63553056}f_{n+7})h+\frac{1}{h^{2}}(-\frac{8020143}{21184352}y_{n}+\frac{4228835}{3026336}y_{n+2}-\frac{5067195}{3026336}y_{n+4}$$
$$+\frac{2077553}{3026336}y_{n+6}-\frac{20444}{662011}y_{n+7})$$

$$y''_{n+6}(x)=(\frac{72576353}{953295840}f_{n+1}+\frac{13449847}{59580990}f_{n+2}+\frac{157080227}{317765280}f_{n+3}+\frac{294914099}{953295840}f_{n+5}-\frac{3185113}{190659168}f_{n+7})h+\frac{1}{h^{2}}(\frac{4308601}{31776528}y_{n}-\frac{690235}{1513168}y_{n+2}+\frac{4106917}{4539504}y_{n+4}$$
$$-\frac{6275827}{4539504}y_{n+6}+\frac{1585544}{1986033}y_{n+7})$$

$$y''_{n+7}(x)=(\frac{363626833}{953295840}f_{n+1}+\frac{3711709}{2648044}f_{n+2}+\frac{541628797}{317765280}f_{n+3}-\frac{803338327}{317765280}f_{n+5}+\frac{272455997}{953295840}f_{n+7})h+\frac{1}{h^{2}}(\frac{15394241}{21184352}y_{n}-\frac{9276493}{3026336}y_{n+2}+\frac{16536181}{3026336}y_{n+4}$$
$$-\frac{18620319}{3026336}y_{n+6}+\frac{2004068}{662011}y_{n+7})$$

$$y''_{n+8}(x)=(\frac{214088981}{317765280}f_{n+1}-\frac{53816779}{59580990}f_{n+2}+\frac{3573747677}{317765280}f_{n+3}+\frac{25241375149}{953295840}f_{n+5}+\frac{1175368127}{317765280}f_{n+7})h+\frac{1}{h^{2}}(\frac{7148709}{10592176}y_{n}+\frac{4531147}{1513168}y_{n+2}-\frac{21748143}{1513168}y_{n+4}$$
$$+\frac{35439985}{1513168}y_{n+6}-\frac{8419352}{662011}y_{n+7})$$

(2.5)

Evaluating the third derivative of the continuous scheme at x_{n+6} , x_{n+8} and further simplification gives the remaining two schemes:

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$$\left.\begin{aligned} f_{n+6} &= \left(\frac{357529}{756584}f_{n+1}+\frac{149827}{94573}f_{n+2}+\frac{1906625}{756584}f_{n+3}-\frac{1262885}{756584}f_{n+5}-\frac{6901}{756584}f_{n+7}\right) \\ &\qquad \frac{+\frac{330375}{378292}y_n-\frac{1279845}{378292}y_{n+2}+\frac{1953525}{378292}y_{n+4}-\frac{1773975}{378292}y_{n+6}+\frac{192480}{94573}y_{n+7}}{h^3} \\ f_{n+8} &= \left(\frac{1326615}{756584}f_{n+1}-\frac{424123}{94573}f_{n+2}+\frac{1906625}{756584}f_{n+3}+\frac{64625365}{756584}f_{n+5}+\frac{5580309}{756584}f_{n+7}\right) \\ &\qquad \frac{+\frac{129150}{94573}y_n+\frac{1203930}{94573}y_{n+2}-\frac{4917150}{94573}y_{n+4}-\frac{7827750}{94573}y_{n+6}-\frac{4243680}{94573}y_{n+7}}{h^3} \end{aligned}\right\} \quad (2.6)$$

The derived block of twenty-four discrete schemes of (2.3), (2.4), (2.5) and (2.6) simultaneously solves third order linear integro-differential equation with the choices of Boole, Simpson 1/3 and Trapezoidal quadrature rules.

3.0 Analysis of the Methods

The basic behavioural properties of the methods are analyzed to establish the efficiency, reliability and validity of the method. These properties include order, error constant, consistence, stability and convergence.

3.1 Order and Error constant of the main schemes and its derivatives

Employing the Taylor series of the main scheme $y_{n+8}(x)$ of (2.3), its first and second derivatives $y'_{n+8}(x)$ and $y''_{n+8}(x)$ of (2.4) and (2.5) respectively by collecting the like terms in term of h, we obtained the error constants as

$$\left.\begin{aligned} c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = \frac{16669}{6620110} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = \frac{56427727}{1021388400} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = \frac{356941087}{1225666080} \end{aligned}\right\} \quad (3.1)$$

By extension, the block of twenty fours discrete schemes in (2.3), (2.4), (2.5) and (2.6) is found to be of uniform order $\rho = 7$ with a varying error constant.

3.2 Consistency of the method

Definition 1: A linear multistep method is consistent if the following conditions hold:

(a) The order is greater than one, i.e. $\rho \geq 1$

(b) $\sum_{i=0}^{k} \alpha_i = 0$

(c) $\rho(1) = p'(1) = \rho''(1) = 0$

(d) $\rho'''(1) = 3!\delta(1)$

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The main scheme $y_{n+8}(x)$ of (2.3) satisfies all the conditions in definition 1, hence the main scheme is consistent and by extension, other schemes are found to be consistent.

3.3 Zero stability

Definition 2: A linear multistep method is said to be zero stable if all the roots of characteristic polynomial $\rho(r)$ satisfying $|r| \leq 1$ are distinct, with the multiplicity of roots $\rho(r)=1$ not exceeding the order of the differential equations the method seeks to solve. Following this definition, the roots of the first characteristic polynomial

$$r^8 - \frac{359296}{94573}r^7 + \frac{449563}{94573}r^6 - \frac{344207}{94573}r^4 + \frac{218969}{94573}r^2 - \frac{59602}{94573} = 0 \quad (3.2)$$

associated with main scheme are:

$$\left.\begin{aligned} r_1 &= 1.228601637 + 0.9108042616I \\ r_2 &= -0.5415564784 + 0.4187409649I \\ r_3 &= -.05749510281 \\ r_4 &= -0.5749510281 - 0.418749649I \\ r_5 &= 1.228601637 - 0,910804261I \\ r_6 &= 1 \\ r_7 &= 1 \\ r_8 &= 1 \end{aligned}\right\} \quad (3.3)$$

Hence, the method is zero stable.

3.4 Region of Absolute Stability of the main scheme

Using the function $\prod(r,H) = \rho(r) - H\delta(r)$.the region of absolute stability is sketched for $H(r) = \frac{\rho(r)}{\delta(r)}$,where $r = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ and presented

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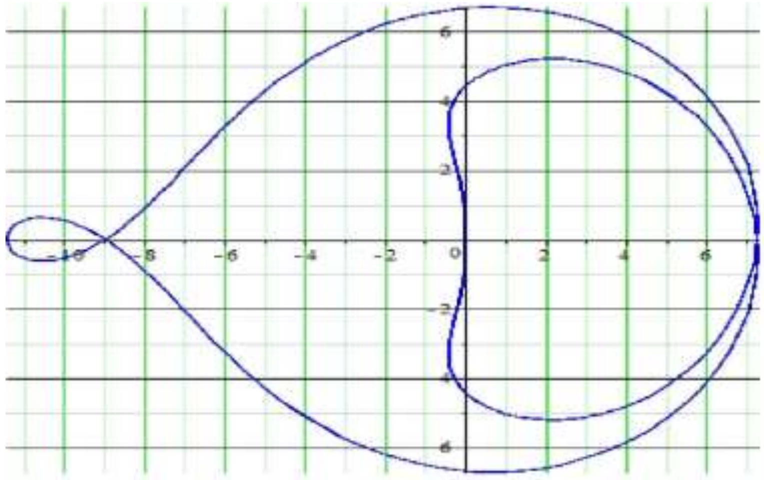

Figure 1: Region of absolute stability of the main scheme

3.5 Convergence

According to [16] and [17], A numerical method is convergent if it is consistent and zero stable. Thus, the proposed method is convergent.

4.0 Numerical Experiment

The selected third order initial value problems of FIDE are solved to support our theoretical discussion on the derived method; the method is implemented with the choice of Boole, Simpson 1/3 and Trapezoidal rules. The experiments are performed with the aid of MAPLE 18 software package.

Problem 1

Consider the linear FIDE $y'''(x) = 6 + x - \int_0^1 xy''(t)dt, y(0) = -1, y'(0) = 1, y''(0) = -2$ with exact solution $y(x) = x^3 - x^2 + x - 1$

Source: [1]

Table 1: Solution to problem 1 using the derived scheme with Boole, Simpson 1/3 and Trapezoidal formula for $n=8, h=0.125$

Problem 2

x	***y*(exact)**	**Boole**	**Simpson 1/3**	**Trapezoidal**	**Error in Boole**	**Error in Simpson 1/3**	**Error in Trapezoidal**
0.125	-0.888671875	-0.888671874	-0.888671875	-0.888671874	1.5000E-09	1.3000E-09	1.3000E-09
0.25	-0.796875000	-0.796874992	-0.796874993	-0.796874993	7.6000E-09	7.4000E-09	7.1000E-09
0.375	-0.712890625	-0.712890607	-0.712890607	-0.712890608	1.8200E-08	1.8000E-08	1.7200E-08
0.5	-0.625000000	-0.624999967	-0.624999967	-0.624999968	3.3200E-08	3.3200E-08	3.1700E-08
0.625	-0.521484375	-0.521484323	-0.521484323	-0.521484325	5.2100E-08	5.2200E-08	5.0000E-08
0.75	-0.390625000	-0.390624926	-0.390624925	-0.390624929	7.4400E-08	7.4700E-08	7.1500E-08
0.875	-0.220703125	-0.220703025	-0.220703024	-0.220703029	1.0000E-07	1.0100E-07	9.6400E-08
1	0.000000000	1.290838E-07	1.300515E-07	1.2457967-07	1.2908E-07	1.3005E-07	1.2458E-07

Consider the LFIDE $y'''(x) = \sin x - x - \int_0^{\frac{\pi}{2}} xty''(t)dt, y(0)=1, y'(0)=0, y''(0)=-1$ with exact solution

$y(x) = \cos x$

Source: [18]

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Table 2: Solution to problem 2 using the derived scheme embedded with Boole, Simpson 1/3 and Trapezoidal quadrature formula for $n = 8, h = 0.1963750$

x	y(exact)	Boole	Simpson 1/3	Trapezoidal	Error in Boole	Error in Simpson 1/3	Error in Trapezoidal
0.1963750	0.9807803133	0.9807803174	0.9807803161	0.9807804675	4.1000E-09	2.8000E-09	1.5420E-07
0.3927500	0.9238600457	0.9238602303	0.9238602109	0.9238626325	1.8460E-07	1.6520E-07	2.5868E-06
0.5891250	0.8314271768	0.8314282616	0.8314281636	0.8314404229	1.0848E-06	9.8680E-07	1.3246E-05
0.7855000	0.7070347682	0.7070383794	0.7070380705	0.7070768158	3.6112E-06	3.3023E-06	4.2048E-05
0.9818750	0.5554643860	0.5554734204	0.5554726672	0.5555672597	9.0344E-06	8.2812E-06	1.0287E-04
1.1782500	0.3825423008	0.3825612884	0.3825597274	0.3827558740	1.8988E-05	1.7427E-05	2.1357E-04
1.3746250	0.1949155292	0.1949509947	0.1949481037	0.1953114892	3.5466E-05	3.2575E-05	3.9596E-04
1.5710000	-2.036732E-04	-1.428475E-07	-1.477782E-04	4.721408-04	6.0826E-05	5.5895E-05	6.7581E-04

Problem 3

Consider the LFIDE $y'''(x) = \sin x - 2x - \cos x + \int_0^{\frac{\pi}{2}} xy''(t)dt, y(0) = y'(0) = 1, y''(0) = -1$ with exact solution $y(x) = \sin x + \cos x$

Source: [19]

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Table 3: Solution to problem 3 using the derived scheme embedded with Boole, Simpson 1/3 and Trapezoidal quadrature formula for $n=8, h=0.392750$

x	y(exact)	Boole	Simpson 1/3	Trapezoidal	Error in Boole	Error in Simpson 1/3	Error in Trapezoidal
0.392750	1.306590520	1.306576382	1.306576201	1.306592118	1.4138-05	1.4319E-05	1.5980E-06
0.785500	1.414213555	1.414154670	1.414151749	1.414406438	5.8885E-05	6.1806E-05	1.9288E-04
1.178250	1.306480279	1.306389398	1.306374585	1.307663957	9.0881E-05	1.0570E-04	1.1837E-02
1.571000	0.999796306	0.999760293	0.999713439	1.003788500	3.6013E-05	8.2868E-05	3.9922E-03
1.963750	0.540863443	0.541074168	0.540959729	0.550908622	2.1072E-04	9.6286E-05	1.0045E-02
2.356500	-0.000432056	0.000351904	0.000114545	0.020744582	7.8396E-04	5.4660E-04	2.1177E-02
2.749250	-0.541661762	-0.539814741	-0.540254543	-0.502034838	1.8470E-04	1.4072E-03	3.9627E-02
3.142000	-1.000407263	-0.996814172	-0.997564528	-0.932363383	3.5931E-04	2.8427E-03	6.8044E-02

Problem 4

Consider the LFIDE $y'''(x)=1-e+e^{x}+\cos x+\int_{0}^{1} y(t)dt, y(0)=y'(0)=y''(0)=1$ with exact solution

$y(x)=e^{x}$

Source: [19]

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Table 4: Solution to problem 4 using the derived scheme embedded with Boole, Simpson 1/3 and Trapezoidal quadrature formula for $n=8, h=0.125$

x	y(exact)	Boole	Simpson 1/3	Trapezoidal	Error in Boole	Error in Simpson 1/3	Error in Trapezoidal
0.125	1.133148453	1.133148450	1.133148449	1.133149207	3.0000E-09	4.0000E-09	7.5400E-07
0.250	1.284025417	1.284025398	1.284025402	1.284031465	1.9000E-08	1.5000E-08	6.0480E-06
0.375	1.454991415	1.454991367	1.454991386	1.455011858	4.8000E-08	2.9000E-08	2.0443E-05
0.500	1.648721271	1.648721181	1.648721229	1.648769771	9.0000E-08	4.2000E-08	4.8500E-05
0.625	1.868245957	1.868245813	1.868245909	1.868340738	1.4400E-07	4.8000E-08	9.4781E-05
0.750	2.117000017	2.116999807	2.116999973	2.117163868	2.1000E-07	4.4000E-08	1.6385E-04
0.875	2.398875294	2.398875005	2.398875271	2.399135561	2.8900E-07	2.3000E-08	2.6027E-04
1.000	2.718281828	2.718281442	2.718281845	2.718670414	3.8600E-07	1.7000E-08	3.8859E-04

Table 5: Maximum Absolute Error (MaxE) of the Derived method with different quadrature rule for Problems 1-4 with $n=8$

Quadrature Formula	MaxE in Exp. 1	MaxE in Exp. 2	MaxE in Exp. 3	MaxE in Exp. 4
Boole	1.2908E-07	6.0826E-05	7.8396E-04	3.8600E-07
Trapezoidal	1.2458E-07	6.7581E-04	6.8044E-02	3.8859E-04
Simpson 1/3	1.3005E-07	5.5895E-05	2.8427E-03	4.8000E-08

Table 6: Maximum Absolute Error (MaxE) of the derived method with different quadrature rule for Experiment 1-4 with $n=32$

Quadrature Formula	MaxE in Exp. 1	MaxE in Exp. 2	MaxE in Exp. 3	MaxE in Exp. 4
Boole	7.5784E-06	5.7806E-03	1.0615E-03	1.4104E-05
Trapezoidal	7.3928E-06	1.0209E-04	5.2435E-03	1.4364E-05
Simpson 1/3	1.4741E-03	6.3469E-05	1.0639E-03	4.2194E-05

5.0 Discussion and Conclusion

In this paper, we derived an 8-step block linear multistep method for solving IVPs in LFIDE. The method was derived using Vieta-Pell-Lucas Polynomials as the Interpolant and the derived method is implemented with Boole, Simpson 1/3 and Trapezoidal quadrature rule. We examined the performance of the method in terms of accuracy and stability and compare them with exact solutions of the problems considered. The new method performed well in terms of accuracy and low error of deviation. The schemes were also compared with each other when embedded with each of the quadrature formula. For example: Table 1 reveals that the derived method when implemented with Trapezoidal rule performed slightly better than when implemented with Simpson 1/3 rules and Boole rules in example 1. While Table 2-4 show that the derived method when implemented with Simpson 1/3 rule is relatively better than when implemented with the other two quadrature formulae. Table 5 & Table 6 further reveal the maximum absolute error in each of the problems as the number of partition increases from $n=8$ to $n=32$.

The experimental results further reveal that the choice of quadrature formulae and the nature of the problems play vital roles in the level and quality of solution obtainable when solving Linear Fredholm Integro differential equation of third order. Results also show higher rate of convergence of the computed solution with exact solution when we have lesser number of partitions. All numerical experiments were carried out with Maple 18 with a precision of 10 decimal places.

6.0 Conflict of Interests

The authors declare no conflict of interests regarding the publication of this paper.

7.0 Acknowledgements

The authors wish to state that all the cited materials are duely referenced.